

Algebraic and combinatorial properties of binomial edge ideals

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Commutative algebra was built in step with algebraic geometry and played an essential role in its development. In the 1950's, homological aspects of modern commutative algebra became a new and important focus of research inspired by the work of Melvin Hochster. In 1975, Richard Stanley proved affirmatively the upper bound conjecture for spheres by using the theory of Cohen-Macaulay rings. This created another new trend of commutative algebra, as it turned out that commutative algebra supplies basic methods in the algebraic study of combinatorics on convex polytopes and simplicial complexes. Stanley was the first to use concepts and techniques from commutative algebra in a systematic way to study simplicial complexes by considering the Hilbert function of Stanley-Reisner rings, whose defining ideals are generated by square-free monomials. Since then, the study of square-free monomial ideals from both the algebraic and combinatorial points of view has become a very active area of research in commutative algebra.

Finite simple graphs are just a special class of simplicial complexes and so the results on Stanley-Reisner ideals can be used to study monomial edge ideals. The study of these ideals was started by Rafael Villarreal in the 1990's and, since then, a lot of mathematicians studied their algebraic properties in terms of the combinatorial properties of the underlying graph. In 2003, in [6], Jürgen Herzog and Takayuki Hibi classified Cohen-Macaulay bipartite graphs and in 2010, in [9], Russ Woodroffe showed that the regularity of the monomial edge ideal of a weakly chordal graph is given by its induced matching number. These two results are particularly important in studying binomial edge ideals of closed graphs.

On the other hand, in the late 1980's, the theory of Gröbner bases and initial ideals came into fashion in many branches of mathematics, for it provided new methods. They have been used not only for computational purposes but also to deduce theoretical results in commutative algebra and combinatorics. For example, based on the fundamental work of Gelfand, Kapranov, Zelevinsky and Sturmfels, the study of regular triangulations of a convex polytope by using suitable initial ideals (far beyond the classical techniques in combinatorics) turned out to be a very successful approach, and, after the pioneering work of Sturmfels, the algebraic properties of determinantal ideals have been explored by considering their initial ideal, which for a suitable monomial order is a square-free monomial ideal and hence is accessible to powerful techniques.

At about the same time, Galligo, Bayer and Stillman observed that generic initial ideals have particularly nice combinatorial structures and provide a basic

tool for the combinatorial and computational study of the minimal free resolution of a graded ideal of the polynomial ring. Algebraic shifting, which was introduced by Kalai and which contributed to the modern development of enumerative combinatorics on simplicial complexes, can be discussed in the frame of generic initial ideals.

Chapter 1 consists on a first course in monomial ideals, simplicial complexes and Gröbner bases. Even if a reader does not want to study monomial edge ideals or binomial edge ideals, these topics are fundamental pre-requisites for anyone who wishes to study Combinatorial Commutative Algebra.

Monomial ideals are introduced in section 1.1. Monomials form a natural K -basis for the polynomial ring $S = K[x_1, \dots, x_n]$, where K is a field. A monomial ideal I also has a K -basis of monomials. As a consequence, a polynomial $f \in S$ belongs to I if and only if all monomials in f with a non-zero coefficient also belong to I . This is one of the reasons why algebraic operations with monomial ideals are easy to perform and one may take advantage of this fact when studying general ideals in S by considering its initial ideal with respect to some monomial order.

Section 1.2 consists in the study of simplicial complexes. Sometimes, when studying graphs, is it a good idea to think about them as simplicial complexes. In fact, for each graph G , we can associate a simplicial complex $\Delta(G)$, called its clique complex, such that there is an obvious correspondence between the faces of G and the cliques of $\Delta(G)$. Since each graph is uniquely determined by its clique complex, then graphs are just a special class of simplicial complexes. To each simplicial complex Δ we associate its Stanley-Reisner ideal I_Δ and say that Δ is a Cohen-Macaulay simplicial complex over K if and only if S/I_Δ is a Cohen-Macaulay ring. In section 1.2 it is shown that a Cohen-Macaulay simplicial complex is strongly connected. This result is used to classify Cohen-Macaulay bipartite graphs in chapter 2.

Gröbner bases are introduced in section 1.3. Given a monomial order $<$ on S , we define the initial ideal $\text{in}_<(I)$ as the monomial ideal in S generated by the monomials $\text{in}_<(f)$ with $f \in I \setminus \{0\}$. A Gröbner basis for I is just a set of non-zero polynomials in I whose initial monomials generate $\text{in}_<(I)$. A Gröbner basis turns out to be a generator set of the corresponding ideal.

Fix a monomial order $<$ on S and let g_1, \dots, g_s be non-zero polynomials. For every non-zero polynomial $f \in S$ we define what is standard expression of f with respect to g_1, \dots, g_s . Standard expressions generalize the division algorithm in $K[x]$. But unlike what happens with the division algorithm in $K[x]$, a standard expression may not be unique.

The remainder is unique in the particular case where g_1, \dots, g_s is a Gröbner basis for the ideal $I = (g_1, \dots, g_s)$. Moreover, a non-zero polynomial $f \in S$ reduces to 0 if and only if $f \in I$.

Usually it is very hard to show that a given generator set of an ideal is a Gröbner basis and so the S -polynomials and Buchberger's criterion provide us a nicer way to do so: a generator set $\{g_1, \dots, g_s\}$ of I is a Gröbner basis for I if and only if all the S -pairs $S(g_i, g_j)$ reduce to 0 with respect to g_1, \dots, g_s . Moreover, the Buchberger's criterion supplies an algorithm to compute a Gröbner basis of a given ideal starting from a system of generators of such ideal, called the Buchberger's algorithm.

Chapter 2 is focused in the study of binomial edge ideals.

In section 2.1, monomial edge ideals are introduced and Cohen-Macaulay bipartite graphs are classified. In fact, a bipartite graph H on (V, V') without isolated vertices is Cohen-Macaulay if and only if there exists labellings $V = \{x_1, \dots, x_n\}$ and $V' = \{y_1, \dots, y_n\}$ such that:

- $\{x_i, y_i\}$ is an edge of H for every $i \in [n]$.
- If $i > j$, then $\{x_i, y_j\}$ is not an edge of H .
- If $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are edges of H , then so is $\{x_i, y_k\}$.

Section 2.2 is an introduction to the study of binomial edge ideals.

Let G be a simple graph on the vertex set $[n]$. We define the binomial edge ideal of G as the ideal of $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ generated by the binomials $x_i y_j - x_j y_i$ where $i < j$ and $\{i, j\}$ is an edge of G .

Binomial edge ideals were introduced for the first time in [7], in 2009. This ideals can be seen as a natural generalization of 2-minors of a $2 \times n$ matrix of indeterminates. The results on minimal prime ideals of a binomial edge ideal apply for the class of conditional independence ideals where a fixed binary variable is independent of a collection of other variables, given the remaining ones. In this case the primary decomposition has a natural statistical interpretation.

A simple graph G is closed if and only if for all edges $\{i, j\}$ and $\{k, l\}$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$ and $\{i, k\} \in E(G)$ if $j = l$.

Let $<$ be the lexicographic order on $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ induced by $x_1 > \dots > x_n > y_1 > \dots > y_n$. Then G is a closed graph if and only if the standard generators of J_G are a Gröbner basis of J_G with respect to $<$.

Let S be a subset of $[n]$ and let $G_1, \dots, G_{c(S)}$ be the connected components of the induced graph of G on $[n] \setminus S$, denoted by $G_{[n] \setminus S}$. For each subset $S \subset [n]$ we define $P_S(G) = (x_i, y_i : i \in S) + J_{\tilde{G}_1} + \dots + J_{\tilde{G}_{c(S)}}$. Using an important result on toric ideals and the Buchberger's algorithm (recall that complete graphs are closed graphs), we find out that $P_S(G)$ is a prime ideal.

While it is true that every minimal prime ideal of J_G has this form, not every such prime ideal is a minimal prime of J_G . For example, $P_\emptyset(G)$ is always a minimal prime of J_G but $P_{[n]}(G)$ is never a minimal prime of J_G for it clearly contains all the remaining $P_S(G)$.

In section 2.2 we determine the minimal prime ideals of J_G . In other words, we determine the minimal sets of the collection $\{P_S(G) : S \subset [n]\}$ and it turns out that $P_S(G)$ is such a minimal set if and only if, for every $i \in S$, $c(S \setminus \{i\}) < c(S)$.

Since $\dim(S/P_S(G)) = n - |S| + c(S)$ for every $S \subset [n]$, knowing that all minimal primes of J_G have the form $P_S(G)$ allow us to compute $\dim(S/J_G)$, and since $P_\emptyset(G)$ is a minimal prime of J_G , it follows that if J_G is unmixed (and in particular if J_G is a Cohen-Macaulay ideal), then $\dim(S/J_G) = n + c$, where c is the number of connected components of G . As a simple consequence of this, one sees that a circle of length n is unmixed or Cohen-Macaulay if and only if $n = 3$.

Similar to what happens with monomial edge ideals, a general classification of Cohen-Macaulay binomial edge ideals seems to be hopeless. However, in [8], in 2010, the Cohen-Macaulayness of binomial edge ideals was studied for two

special classes of graphs: closed graphs and chordal graphs such that any two maximal cliques intersect in at most one vertex.

In section 2.3 we assume that G is a connected chordal graph such that any two maximal cliques intersect in at most one vertex and consider a leaf order F_1, \dots, F_r on the facets of $\Delta(G)$. If $r = 1$, then G is a complete graph. It was shown in [1] that J_G is a Cohen-Macaulay ideal when G is complete.

So assume that $r > 1$. Then there exists a unique vertex $i \in F_r$ which also belongs to some other facet of $\Delta(G)$. Let F_{t_1}, \dots, F_{t_q} be the facets of $\Delta(G)$ which intersect the leaf F_r in the vertex i . Consider the decomposition $J_G = Q_1 \cap Q_2$, where Q_1 is the intersection of all minimal primes $P_S(G)$ with $i \notin S$ and Q_2 is the intersection of all minimal primes $P_S(G)$ with $i \in S$. On one hand, $Q_1 = J_{G'}$, where G' is obtained from G by replacing the facets $F_{t_1}, \dots, F_{t_q}, F_r$ by a clique on their union. Moreover, G' is also a connected chordal graph such that any two maximal cliques intersect in at most one vertex. On the other hand, $S/Q_2 \cong S_i/J_{G''}$, where G'' is the restriction of G to the vertex set $[n] \setminus \{i\}$. Moreover, G'' is a chordal graph with $q + 1$ connected components such that any two maximal cliques intersect in at most one vertex. By last, $S/(Q_1 + Q_2) \cong S_i/J_H$, here H is the restriction of G to the vertex set $[n] \setminus \{i\}$. Again, H is a connected chordal graph such that any two maximal cliques intersect in at most one vertex.

The decomposition $J_G = Q_1 \cap Q_2$ yields the following exact sequence of S -modules:

$$0 \longrightarrow \frac{S}{J_G} \longrightarrow \frac{S}{Q_1} \oplus \frac{S}{Q_2} \longrightarrow \frac{S}{Q_1 + Q_2} \longrightarrow 0.$$

Using this exact sequence and induction it is shown that if G is a chordal graph such that any two maximal cliques intersect in at most one vertex, then $\text{depth}(S/J_G) = n + c$, where n is the number of vertices of G and c is the number of connected components of G . Since $\dim(S/J_G) \geq n + c$, it follows that J_G is unmixed if and only if it is Cohen-Macaulay. Hence, to determine for which chordal graphs G such that any two maximal cliques intersect in at most one vertex J_G is a Cohen-Macaulay ideal, we just need to determine for which such graphs J_G is an unmixed ideal. We show that J_G is unmixed if and only if every vertex of G lies in at most two distinct maximal cliques.

Another class of graphs whose binomial edge ideals are Cohen-Macaulay if and only if they are unmixed is the class of closed graphs. We study the Cohen-Macaulayness of these ideals in section 2.4.

If G is closed, then $\text{in}_<(J_G)$ is the monomial edge ideal of the bipartite graph $H = \text{in}_<(G)$ on $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ such that $\{x_i, y_j\}$ is an edge of H if and only if $i < j$ and $\{i, j\}$ is an edge in G . For an arbitrary ideal $I \subset S$, we know that I is a Cohen-Macaulay ideal whenever $\text{in}_<(I)$ is also Cohen-Macaulay. Using this result we get that J_G is a Cohen-Macaulay ideal whenever H is a Cohen-Macaulay graph. In [8], Ene, Herzog and Hibi show that in fact J_G is a Cohen-Macaulay ideal if and only if H is a Cohen-Macaulay graph. And in section 2.1 a classification of Cohen-Macaulay bipartite graphs is presented.

Chapter 3 consists in the study of the Castelnuovo-Mumford regularity of binomial edge ideals.

Castelnuovo-Mumford regularity is one of the most fundamental invariants in Commutative Algebra and Algebraic Geometry. In fact, already in the late 19th

century this invariant was present, a long time before it was properly defined. One of its first hidden appearances may be found in Castelnuovo's work.

However, a proper definition of Castelnuovo-Mumford regularity was only given in 1966 by Mumford, who called it Castelnuovo regularity. In fact, Mumford defined the notion of being m -regular in the sense of Castelnuovo for a coherent sheaf of ideals over a projective space and a given integer m . More precisely, a sheaf of ideals over a projective space is called m -regular if, for every $i > 0$, the i -th Serre cohomology group of the $(m - i)$ -fold twist of this sheaf vanishes. The smallest m such that the sheaf of ideals in question is m -regular is what today is usually called the Castelnuovo-Mumford regularity of such sheaf.

Castelnuovo-Mumford regularity also found much interest in Commutative Algebra. In 1982, Ooishi defined the Castelnuovo-Mumford regularity of a graded module in terms of certain local cohomology modules. His definition essentially corresponds to Mumford's definition via the Serre-Grothendieck correspondence between local cohomology and sheaf cohomology. In 1984, Eisenbud and Goto made explicit the link between this "algebraic" Castelnuovo-Mumford regularity of a graded module over a polynomial ring and its minimal free resolution.

As Russ Woodrooffe wrote in [9], the Castelnuovo-Mumford regularity of an ideal I is one of the main measures of the complexity of I . Woodrooffe also showed that the regularity of a weakly chordal graph is given by its induced matching number. The same way Woodrooffe studied the regularity of monomial edge ideals, recently some mathematicians started studying the regularity of binomial edge ideals.

In 2012, in [11], Matsuda and Murai showed that, for a simple graph G on n vertices, one has $l \leq \text{reg}(S/J_G) \leq n - 1$, where l is the length of the longest induced path in G . Moreover, they conjectured that the equality $\text{reg}(S/J_G) = n - 1$ can only be reached when G is a path. Later in the same year, in [10], Madani and Kiani conjectured that if G is a graph with r maximal cliques, then $\text{reg}(S/J_G) \leq r$. In 2013, in [12], Viviana Ene and Andrei Zarojanu showed that the equality $\text{reg}(S/J_G) = l$ is reached when G is a closed graph. With respect to the class of chordal graphs such that any two maximal cliques intersect in at most one vertex, Ene and Zarojanu showed that this class of graphs satisfy both the Madani-Kiani and the Matsuda-Murai conjectures.

In section 3.2 we determine $\text{reg}(S/J_G)$ when G is a closed graph. For such a graph, let $H = \text{in}_<(G)$. For an arbitrary ideal $I \subset S$, we know that $\text{reg}(S/I) \leq \text{reg}(S/\text{in}_<(I))$. Using this result we get that $\text{reg}(S/J_G) \leq \text{reg}(S/I(H))$. We show that H is a weakly chordal bipartite graph whose induced matching number is equal to the length l of the longest induced path in G . Hence, using a result in [9], we get that $\text{reg}(S/J_G) \leq l$. Combining this inequality with the inequality $\text{reg}(S/J_G) \geq l$ shown by Matsuda and Murai, we get $\text{reg}(S/J_G) = l$, as desired.

In section 3.3, the last section of this dissertation, we consider once again chordal graphs such that any two distinct maximal cliques intersect in at most one vertex. If G is such a connected graph, we consider again the exact sequence

$$0 \longrightarrow \frac{S}{J_G} \longrightarrow \frac{S}{Q_1} \oplus \frac{S}{Q_2} \longrightarrow \frac{S}{Q_1 + Q_2} \longrightarrow 0.$$

Using this exact sequence and induction it is shown that the Madani-Kiani conjecture holds for this class of graphs.

We end by showing that the Matsuda-Murai conjecture also holds for this class of graphs. In fact, if G is a connected graph such that $\text{reg}(S/J_G) = n - 1$, then G has at least $n - 1$ maximal cliques and so it must be a tree. Using induction and the same exact sequence as before we get that G is a path.

While in this dissertation the Matsuda-Murai conjecture was meant to be presented as a conjecture, it happened to be fully shown this year, on April 6th, in [13] by Madani and Kiani. Since the partial proof for chordal graphs such that any two maximal cliques intersect in at most one vertex is an interesting proof which uses some nice results on the combinatorial properties of these graphs, I decided to keep it in the dissertation, not forgetting to give a reference for the full proof of the Matsuda-Murai conjecture.

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